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A LOWER BOUND FOR THE RECOGNITION OF DIGRAPH PROPERTIES

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The complexity of a digraph property is the number of entries of the adjacency matrix which must be examined by a decision tree algorithm to recognize the property in the worst case. Aanderaa and Rosenberg conjectured that there is a constant ε such that every digraph property which is monotone (not destroyed by the deletion of edges) and nontrivial (holds for some but not all digraphs) has complexity at least εv^2 where v is the number of nodes in the digraph. This conjecture was proved by Rivest and Vuillemin and a lower bound of $v^2/4-o(v^2)$ was subsequently found by Kahn, Saks, and Sturtevant. Here we show a lower bound of $v^2/2-o(v^2)$. We also prove that a certain class of monotone, nontrivial bipartite digraph properties is evasive (requires that every entry in the adjacency matrix be examined in the worst case).

1. Introduction

Suppose we would like to determine whether an unknown digraph on nodes $V=\{1, 2, ..., v\}$ has, for example, a node with no incoming edges, and we can obtain information only by asking questions of the form "Is edge (i,j) in the graph?". In the decision tree model considered here, the choice of question may depend on the information gained so far and the complexity of a problem is the number of questions that must be asked in the worst case.

A *digraph* with nodes $V=\{1, 2, ..., v\}$ may be viewed as a subset of the edges on V, i.e., $\{(i,j)|i,j\in V, i\neq j\}$. A collection of such digraphs is called a *digraph property* provided that it is invariant under renumbering of the nodes. The definition of *graph property* is the same, except that edges consist of 2-element subsets of V, instead of ordered pairs.

A graph or digraph property is *evasive* if every edge in E must be asked about in the worst case. It is *monotone* (decreasing) if it is not destroyed by the removal of edges. It is *nontrivial* if it holds for some, but not all graphs.

In 1973, S. Aanderaa and R. L. Rosenberg proposed the following:

Aanderaa—Rosenberg Conjecture [7]. *There is a constant $\varepsilon>0$ such that every nontrivial monotone digraph property on v nodes has complexity at least εv^2.*

The Aanderaa—Rosenberg Conjecture was proved by Rivest and Vuillemin [6]. Their value of $\varepsilon=1/16$ was improved by Kleitman and Kwiatkowski [3] to $\varepsilon=1/9$. These results were actually shown for graph, rather than digraph properties. If we let M_v and M_v^D denote, respectively, the minimum complexities of non-

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trivial monotone graph and digraph properties on v nodes, the bound for M_v^D follows from the observation that $M_v \leqq M_v^D \leqq 2M_v$.

In 1984, Kahn, Saks, and Sturtevant showed an improved asymptotic bound for ε:

$$M_v^D \geqq M_v \geqq v^2/4+o(v^2).$$

In this paper, we will improve the asymptotic bound for M_v^D by showing that

$$M_v^D \geqq v^2/2+o(v^2).$$

Thus, for every nontrivial monotone graph or digraph property almost half the edges must be asked about in the worst case.

The following well-known conjecture is still open:

Conjecture. *Every nontrivial monotone graph or digraph property is evasive.*

(See [2] for references to related results.)

In the next section, we review the topological approach developed in [2] which will be used here. In Section 3, we prove evasiveness for a special case of nontrivial monotone bipartite digraph properties. This result is used in Section 4 to prove the following relationship, which is analogous to a recursion proved for undirected graph properties in [3]:

Lemma 4.1. $M_v^D \geqq \min(M_{v-1}^D, 2p^\alpha(v-p^\alpha))$, *where* p^α *is the minimum prime power greater than* $v/2$.

We also use the following result:

Prime Power Theorem [2]. *Every monotone, nontrivial graph or digraph property on* v *nodes where* v *is a prime power is evasive.*

Combining Lemma 4.1 with the Prime Power Theorem, we obtain the following theorem, which is our main result.

Theorem. *Any nontrivial monotone digraph property on* v *nodes has complexity* $\geqq v^2/2+o(v^2)$.

Proof. From the lemma and the Prime Power Theorem, we see that $M_v^D \geqq 2p^\alpha(q^\beta+1-p^\alpha)$ where q^β is the largest prime power less than v and p^α is the smallest prime power greater than $v/2$. The result now follows from an implication of the prime number theorem [1], that there is a function $\delta(x)=o(x)$ such that for all x there is a prime between x and $x+\delta(x)$. ∎

2. Topological approach

The notion of a graph property generalizes to any set property. A *set property* is a collection F of F of subsets of $X=\{x_1, x_2, \ldots, x_n\}$.

Then the problem of computing F is as follows: given an input set S, a fixed but unknown subset of X, determine whether $S \in F$ by asking questions of the form "Is $x_i \in S$?" A set property F is *evasive* if n questions must be asked in the worst case. The definitions of monotone and nontrivial can be naturally extended from graph properties to any set property.

Kahn, Saks, Sturtevant [2] developed a technique for proving evasiveness by observing that results from algebraic topology could be applied. In particular, they noted:

Proposition 2.1 [2]. *If P is a monotone set property then $P \setminus \{\emptyset\}$ is an (abstract simplicial) complex (which we call the "complex P").*

This follows from the definition. An *abstract simplicial complex* is a collection Δ of finite non-empty sets, such that if A is an element of Δ then so is every non-empty subset of A.

Each subset of a complex is called a *face*. The dimension of a face A is $|A|-1$. Faces of dimension 0 are called *vertices*. The union of the vertices in Δ is called the *vertex set*. (See [4].)

A *free face* is a non-maximal face which is contained in only one maximal face. An *elementary collapse* of Δ is the process of removing from Δ some free face, together with all faces containing it. We say Δ is *collapsible* if Δ can be reduced to a vertex by a sequence of elementary collapses.

Lemma 2.2 [2]. *If a monotone property F is nonevasive, then the complex F is collapsible.*

The proof is by induction on the number of vertices.

We will find it useful to characterize a complex by studying group actions on the complex.

A group G *acts on* a set X if there is a mapping from $G \times X \to X$ such that the following is true for all $g, g' \in G$ and $x \in X$. (The image of (g, x) for $g \in G$, $x \in X$, is denoted by gx.):

(i) $g(g'x) = (gg')x$, and

(ii) $ex = x$, where e is the identity element of G.

If S is any subset of X such that for all $s \in S$ and all $g \in G$, $gs \in S$, then we say G *acts on* S or S is *G-invariant*.

The action of G on X induces an action on the set of all subsets of X, e.g., $g\{x_a, x_b, \ldots, x_z\} = \{gx_a, gx_b, \ldots, gx_z\}$. Then G *acts on* (or *preserves*) any collection C of subsets of X if for all $S \in C$ and all $g \in G$, $gS \in C$.

Let G be a group which acts on X and preserves Δ. From the induced action of G on Δ, we can define the complex of *fixed points* of Δ denoted Δ_G as:

1. The vertex set Y of $\Delta_G = \{y_A | A$ is a minimal G-invariant face of $\Delta\}$.

2. The faces of Δ_G are all subsets of $Y = \{y_A, y_B, \ldots, y_Z\}$ such that $A \cup B \cup \ldots \cup Z \in \Delta$.

Let Δ be a complex with vertex set X, $|X| = n$, and let f_d be the number of faces of dimension d. Then the *Euler characteristic* of Δ is:

$$\sum_{d=0}^{n-1} (-1)^d f_d.$$

We are now ready to state a result from algebraic topology, which provides the main tool used in this paper.

Lemma 2.3 [5]. *Let G be a group acting on a collapsible complex Δ and let p be prime. If G is cyclic or G has a normal p-subgroup H such that G/H is cyclic, then the Euler characteristic of Δ_G is* 1.

Oliver proves the lemma for complexes which satisfy even weaker conditions. See [5] and [8].

From Lemmas 2.1, 2.2, and 2.3, we have:

Lemma 2.4. *If the complex P for a set property is invariant under a group G which is cyclic or has a normal p-subgroup H such that G/H is cyclic and p is prime, and the Euler characteristic of $\Delta_G \neq 1$, then the set property is evasive.*

3. Bipartite digraphs

Let $V=\{v_1, v_2, \ldots, v_m\}$ and $W=\{w_1, w_2, \ldots, w_n\}$. A *bipartite graph property* is a collection P of subsets of $V\times W$ which is invariant under permutations of V and of W. A *bipartite digraph property* is similarly defined for subsets of $(V\times W)\cup(W\times V)$.

In 1986, Yao showed that every nontrivial monotone bipartite graph property is evasive [9]. The analogous statement for bipartite digraph properties is false. For example, the property of having no edges from V to W can be computed by an algorithm which asks only about the edges from V to W. It need never consider the edges from W to V. We can, however, prove evasiveness for a restricted class of bipartite digraph properties.

Theorem 3.1. *A nontrivial monotone bipartite digraph property P on V, W with $|W|=p^\alpha$, p prime, is evasive if either:*

$$\text{(i)} \quad V\times W\in P \quad \text{and} \quad W\times V\in P,$$

or

$$\text{(ii)} \quad V\times W\notin P \quad \text{and} \quad W\times V\notin P.$$

Proof. Let G be the group generated by two permutations τ and σ of $V\cup W$, where, for all i:

$$\tau(v_i) = v_i,$$

$$\tau(w_i) = w_{i+1 \bmod p^\alpha};$$

$$\sigma(v_i) = v_{i+1 \bmod |V|},$$

$$\sigma(w_i) = w_i.$$

The action of G on V and W induces an action on the edges between V and W, which in turn induces an action on P, since P is invariant under permutations of V and of W. Since τ and σ commute, G has a normal p-subgroup $\langle\tau\rangle$ such that $G/\langle\tau\rangle=\langle\sigma\rangle$ is cyclic.

Any G-invariant face of the complex P which contains one edge from V to W (resp., W to V) must contain all of $V\times W$ (resp., $W\times V$). Then the possible G-invariant faces are $V\times W$, $W\times V$, and $(V\times W)\cup(W\times V)$. The last is not in P

since P is nontrivia'. Then, since either the first two are both in P or both not in P, the Euler characteristic of $P_G=0$ or $2\neq 1$. Hence, P must be evasive.

Remark. The theorem can be extended to the cases where $|W|=p^\alpha q^\beta$ and $|W|==2p^\alpha q^\beta$, for p and q primes. We sketch the proof for $|W|=p^\alpha q^\beta$.

We use a more general lemma of Oliver's:

If G acts on a collapsible complex Δ and G has a normal subgroup G' with G/G' of q-power order, and G' has a normal p-subgroup H with G'/H cyclic, for p and q primes, then the Euler characteristic of $\Delta_G \equiv 1$ mod q.

We denote the elements of W by $w_{i,j}$, where $0\leqq i<p^\alpha$ and $0\leqq j<q^\beta$. Then we let G be the group generated by three permutations σ, η, and θ of $V\cup W$, where σ is defined as above, and for all i and j:

$$\eta(v_i) = v_i,$$

$$\eta(w_{i,j}) = w_{i,j+1 \bmod q^\beta};$$

$$\theta(v_i) = v_i,$$

$$\theta(w_{i,j}) = w_{i+1 \bmod p^\alpha, j}.$$

Then G satisfies the conditions of the lemma stated above. The G-invariant faces are the same as before, and the Euler characteristic of $P_G=0$ or $2\neq 1$ *mod q*.

4. The recursive formula

The proof of the next result relies on a technique used by Kleitman and Kwiatkowski in [3]. If P is a digraph property, we may lower bound the complexity of P by restricting our attention to those digraphs which contain all edges in some set Y, thought of as the subset of edges "known to be in the graph" and no edges from another set N, thought of as the subset of edges "known to be absent from the graph". Then, if Y is in P, we let P' be the collection of subsets of "unknown" edges whose union with Y is in P. We call P' the *set property induced by Y and N.* Any algorithm to determine whether these digraphs are in P must determine whether the unknown edges are in P'. Hence, the complexity of P is at least the complexity of P'.

Lemma 4.1. *$M_v^D \geqq \min\left(M_{v-1}^D, 2p^\alpha(v-p^\alpha)\right)$ where p^α is the minimum prime power greater than $v/2$.*

Proof. Let V' denote the set $\{2, 3, \dots, n\}$, and, for any set of nodes V, let $\hat{V}$ denote the set of all edges on V.

Case (i): The "bidirected star"$=\{(1, i)|$ for all $i\in V'\}\cup\{(i, 1)|$ for all $i\in V'\}\in P$.

Then let $Y=$the bidirected star and $N=\varnothing$. The induced set property P' is a monotone digraph property on $v-1$ nodes and it is nontrivial because $\varnothing\in P'$ and $\hat{V}'\notin P'$ (since P is nontrivial). Thus $M_v^D\geqq M_{v-1}^D$.
Case (ii): $\hat{V}'\notin P$.

Then let $Y=\varnothing$ and $N=$the bidirected star. The induced set property P' is a monotone digraph property on $v-1$ nodes. It is nontrivial because $\varnothing\in P'$ and $\hat{V}'\notin P$. Thus $M_v^D\geqq M_{v-1}^D$.

For the remaining cases, the bidirected star $\notin P$ *and* $\hat{V}'\in P$.

We partition the v nodes into set B with p^α nodes and set A with the remaining nodes. Since $\hat{V}'\in P$, we have $\hat{A}\in P$ and $\hat{B}\in P$.

Case (iii): $\hat{A}\cup\hat{B}\in P$.
There are three subcases:

Case (iiia): $\hat{A}\cup\hat{B}\cup(A\times B)\notin P$ *and* $\hat{A}\cup\hat{B}\cup(B\times A)\notin P$.
Let $Y=\hat{A}\cup\hat{B}$ and $N=\varnothing$. Then P' is a monotone bipartite digraph property on A, B with $|B|$ a prime power. It is nontrivial becauses P is, and it contains neither $A\times B$ nor $B\times A$. By Theorem 3.1, P' is evasive. Hence the complexity of P is at least the complexity of $P'=2p^\alpha(v-p_\alpha)$.

Case (iiib): $L=\hat{A}\cup\hat{B}\cup(A\times B)\in P$.
Claim: $\hat{A}\cup(A\times B)\in P$ and $\hat{A}\cup(B\times A)\in P$.
Proof of Claim: Since P is closed under taking subsets, $L\in P$ implies that $\hat{A}\cup(A\times B)\in P$.
Let B' be a subset of B containing $v-p^\alpha$ nodes. Note that $\hat{B}\subseteqq L$ implies that $\hat{B}'\subseteqq L$ and that $(B-B')\times B'\subseteqq L$. Also, we have $A\times B'\subseteqq L$, so $\hat{B}'\cup\cup(((B-B')\cup A)\times B')\subseteqq L$. Since L is in P, any digraph isomorphic to L is in P. In particular, if we map the nodes in B' to A and the nodes in $(B-B')\cup A$ to B, we have that $\hat{A}\cup(B\times A)\in P$.
Now, let $Y=\hat{A}$ and $N=\hat{B}$. Then the induced set property P' is a monotone, bipartite digraph property on A, B with $|B|$ a prime power and $A\times B$ and $B\times A$ are both in P'. The complete bipartite digraph $\notin P'$; otherwise there would be a bidirected star on a node in A. Since $\varnothing\in P'$, P' is nontrivial. Hence, by Theorem 3.1, P' is evasive and the complexity of $P\geqq 2p^\alpha(v-p^\alpha)$.

Case (iiic): $\hat{A}\cup\hat{B}\cup(B\times A)\in P$.
Similar to Case *iiib*.

Case (iv): $\hat{A}\in P$ *and* $\hat{B}\in P$ *but* $\hat{A}\cup\hat{B}\notin P$.
There are three subcases:

Case (iva): $\hat{A}\cup(A\times B)\notin P$ *and* $\hat{A}\cup(B\times A)\notin P$.
Let $Y=\hat{A}$ and $N=\hat{B}$. Then P' is a monotone, bipartite digraph on A, B which satisfies the conditions of Lemma 3.1. (It is nontrivial by the same argument as in Case *iiib*.) Hence, the complexity of $P\geqq 2p^\alpha(v-p^\alpha)$.

Case (ivb): $\hat{A}\cup(A\times B)\in P$.
Let $Y=\hat{A}\cup(A\times B)$ and $N=\varnothing$.
Claim: The induced property P' is evasive.

Proof. We will use a technique from [2].
Identify the nodes in B with $GF(p^\alpha)$, the field with p^α elements which has a cyclic multiplicative group. Then the group G of automorphisms$=$ $=\{x\to ax+b$: $a, b\in GF(p^\alpha)$, $a\neq 0\}$ acts on the edges in B and the edges in $(B\times A)$ by permuting the nodes in B. Since P is invariant under permutation of nodes, and

the nodes in B are isomorphic to each other in the graph given by the "known" edges, P' is invariant under this action.

G has a normal p-group $H=\{x\to x+b\colon b\in GF(p^\alpha)\}=GF(p^\alpha)$ and $G/H= =GF(p^\alpha)^*$, a cyclic multiplicative group.

Since G maps any two nodes in B to any two other nodes, G maps any edge in $\hat{B}$ to any other edge in $\hat{B}$. Therefore, a G-invariant set must contain either all edges in $\hat{B}$ or no edges in $\hat{B}$. Similarly, G maps any edge (b_i, a_j) to any other edge from a node in B to a_j. Thus, any G-invariant set which contains (b_i, a_j) must contain $B\times\{a_j\}$.

By assumption, $\hat{B}\notin P'$ and a bidirected star $\notin P$, which implies that $B\times\{a_j\}\notin P'$. We conclude that P' has no G-invariant faces. Then the Euler characteristic of $P_G=0\neq 1$. By Lemma 2.4, P' is evasive.

Hence the complexity of $P\geqq$ the complexity of $P'=p^\alpha(v-p^\alpha)+p^\alpha(p^\alpha-1)\geqq \geqq 2p^\alpha(v-p^\alpha)$.

Case (ivc): $\hat{A}\cup(B\times A)\in P$.

Similar to Case *ivb*.

We conclude that $M_v^D\geqq\min\left(M_{v-1}^D, 2p^\alpha(v-p^\alpha)\right)$.

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References

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, 1938.

[2] J. Kahn, M. Saks and D. Sturtevant, A topological approach to evasiveness, *Combinatorica*, 4 (4) (1984), 297—306.

[3] D. J. Kleitman and D. J. Kwiatkowski, Further Results on the Aanderaa—Rosenberg Conjecture, *J. Combinatorial Theory*, **B 28** (1980), 85—95.

[4] J. Munkres, *Elements of Algebraic Topology* (1984).

[5] R. Oliver, Fixed-point sets of group actions on finite acyclic complexes, *Comment. Math. Helv.*, 50 (1975), 155—177.

[6] R. Rivest and S. Vuillemin, On recognizing graph properties from adjacency matrices, *Theor. Comp. Sci.*, 3 (1976), 371—384.

[7] A. L. Rosenberg, On the time required to recognize properties of graphs: A problem, *SIG ACT News*, 5 (4) (1973), 15—16.

[8] P. A. Smith, Fixed point theorems for periodic transformations, *Amer. J. of Math.*, 63 (1941), 1—8.

[9] A. Yao, Monotone Bipartite Graph Properties Are Evasive, *SIAM J. on Computing*, 17 (1986), 517—520.

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